Differential Equations

Exact and Non- Exact Differential Equations

Solutions of Exercises

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Note:

Definition. The expression $M(x,y)dx + N(x,y)dy \ldots (*)$ is called an exact differential in a domain D if there exists a function F of two variables such that the this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is, expression (*) is an excat differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and} \frac{\partial F(x,y)}{\partial y} = N(x,y)\ \forall (x,y) \in D.$$

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation $M(x,y)dx + N(x,y)dy = 0$ is called an exact diffrential equation. Moreover, given the standard form of ordinary differential equation $M(x,y)dx + N(x,y)dy = 0$. The differential equation is said to be exact if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

Part 1. (In exercises 1-10 determine whether or not each of the given equation is exact; solve those that are exact).

1. $(3x + 2y)dx + (2x + y)dy = 0.$

   Solution:

   Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y) = 3x + 2y, N(x,y) = 2x + y,$$

$$\frac{\partial M(x,y)}{\partial y} = 2, \frac{\partial N(x,y)}{\partial x} = 2.$$

   Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 2$$

   we can conclude that the differential equation $(3x + 2y)dx + (2x + y)dy = 0$ is exact differential equation. Furthermore, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x + 2y \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x + y.$$

   From the first of these,

$$F(x,y) = \int M(x,y)\partial x + h(y) = \int (3x + 2y)\partial x + h(y) = \frac{3}{2}x^2 + 2xy + h(y).$$

   Then

$$\frac{\partial F(x,y)}{\partial y} = 2x + \frac{dh(y)}{dy}$$

   But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x + y.$$

   Thus

$$2x + y = 2x + \frac{dh(y)}{dy} \Leftrightarrow y = \frac{dh(y)}{dy}.$$

Thus $h(y) = \frac{1}{2}y^2 + C_0$, where C_0 is an arbitrary constant, and so

$$F(x, y) = \frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 + C_0.$$

Hence a one- parameter family of solution is $F(x, y) = C_1$, or

$$\frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$\frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(3x + 2y)dx + (2x + y)dy = 0$ is $\frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 = C$.

2. $(y^2 + 3)dx + (2xy - 4)dy = 0$.

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x, y) = y^2 + 3, N(x, y) = 2xy - 4,$$

$$\frac{\partial M(x, y)}{\partial y} = 2y, \frac{\partial N(x, y)}{\partial x} = 2y.$$

Since

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 2y$$

we can conclude that the differential equation $(y^2 + 3)dx + (2xy - 4)dy = 0$ is exact differential equation. Afterwards, we must find F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) = y^2 + 3 \text{ and } \frac{\partial F(x, y)}{\partial y} = N(x, y) = 2xy - 4.$$

From the first of these,

$$F(x, y) = \int M(x, y)\partial x + h(y) = \int (y^2 + 3\,)\partial x + h(y) = xy^2 + 3x + h(y).$$

Then

$$\frac{\partial F(x, y)}{\partial y} = 2xy + \frac{dh(y)}{dy}$$

But we must have

$$\frac{\partial F(x, y)}{\partial y} = N(x, y) = 2xy - 4.$$

Thus

$$2xy - 4 = 2xy + \frac{dh(y)}{dy} \Leftrightarrow -4 = \frac{dh(y)}{dy}.$$

Thus $h(y) = -4y + C_0$, where C_0 is an arbitrary constant, and so

$$F(x, y) = xy^2 + 3x + -4y + C_0.$$

Hence a one- parameter family of solution is $F(x, y) = C_1$, or

$$xy^2 + 3x + -4y + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$xy^2 + 3x + -4y = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(y^2+3)dx+(2xy-4)dy=0$ is $xy^2+3x+-4y=C$.

3. $(2xy+1)dx+(x^2+4y)dy=0$.

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y)=2xy+1, N(x,y)=x^2+4y,$$

$$\frac{\partial M(x,y)}{\partial y}=2x, \frac{\partial N(x,y)}{\partial x}=2x.$$

Since

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}=2x$$

we can conclude that the differential equation $(2xy+1)dx+(x^2+4y)dy=0$ is exact differential equation. Afterwards, we must find F such that

$$\frac{\partial F(x,y)}{\partial x}=M(x,y)=2xy+1 \text{ and } \frac{\partial F(x,y)}{\partial y}=N(x,y)=x^2+4y.$$

From the first of these,

$$F(x,y)=\int M(x,y)\partial x+h(y)=\int (2xy+1\ )\partial x+h(y)=x^2y+x+h(y).$$

Then

$$\frac{\partial F(x,y)}{\partial y}=x^2+\frac{dh(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y}=N(x,y)=x^2+4y.$$

Thus

$$x^2+4y=x^2+\frac{dh(y)}{dy} \Leftrightarrow 4y=\frac{dh(y)}{dy}.$$

Thus $h(y)=2y^2+C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y)=x^2y+x+2y^2+C_0.$$

Hence a one- parameter family of solution is $F(x,y)=C_1$, or

$$x^2y+x+2y^2+C_0=C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$x^2y+x+2y^2=C$$

where $C=C_1-C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(2xy+1)dx+(x^2+4y)dy=0$ is $x^2y+x+2y^2=C$.

4. $(3x^2y+2)dx-(x^3+y)dy=0$.

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y)=3x^2y, N(x,y)=-x^3-y,$$

$$\frac{\partial M(x,y)}{\partial y}=3x^2, \frac{\partial N(x,y)}{\partial x}=-3x^2.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = 3x^2 \neq \frac{\partial N(x,y)}{\partial x} = -3x^2$$

we can conclude that the differential equation $(3x^2y + 2)dx - (x^3 + y)dy = 0$ is not exact (non- exact) differential equation.

5. $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y) = 6xy + 2y^2 - 5, N(x,y) = 3x^2 + 4xy - 6,$$

$$\frac{\partial M(x,y)}{\partial y} = 6x + 4y, \frac{\partial N(x,y)}{\partial x} = 6x + 4y.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 6x + 4y$$

we can conclude that the differential equation

$(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$ is exact differential equation.

Moreover, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 6xy + 2y^2 - 5 \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = 3x^2 + 4xy - 6.$$

From the first of these,

$$\begin{aligned} F(x,y) &= \int M(x,y)\partial x + h(y) \\ &= \int (6xy + 2y^2 - 5)\partial x + h(y) \\ &= 3x^2y + 2xy^2 - 5x + h(y) \end{aligned}$$

Then

$$\frac{\partial F(x,y)}{\partial y} = 3x^2 + 4xy + \frac{dh(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 3x^2 + 4xy - 6.$$

Thus

$$3x^2 + 4xy - 6 = 3x^2 + 4xy + \frac{dh(y)}{dy} \Leftrightarrow -6 = \frac{dh(y)}{dy}.$$

Thus $h(y) = -6y + C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = 3x^2y + 2xy^2 - 5x - 6y + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$3x^2y + 2xy^2 - 5x - 6y + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$3x^2y + 2xy^2 - 5x - 6y = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$ is $3x^2y + 2xy^2 - 5x - 6y = C$.

6. $(\theta^2 + 1)\cos r \, dr + 2\theta \sin r \, d\theta = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(r,\theta) = (\theta^2+1)\cos r\,, N(r,\theta) = 2\theta\sin r,$$
$$\frac{\partial M(r,\theta)}{\partial\theta} = 2\theta\cos r\,, \frac{\partial N(r,\theta)}{\partial r} = 2\theta\cos r.$$

Since

$$\frac{\partial M(r,\theta)}{\partial\theta} = \frac{\partial N(r,\theta)}{\partial r} = 2\theta\cos r$$

we can conclude that the differential equation $(\theta^2+1)\cos r\,dr + 2\theta\sin r\,d\theta = 0$ is exact differential equation. Moreover, we must find F such that

$$\frac{\partial F(r,\theta)}{\partial r} = M(r,\theta) = (\theta^2+1)\cos r \text{ and } \frac{\partial F(r,\theta)}{\partial\theta} = N(r,\theta) = 2\theta\sin r$$

From the first of these,

$$\begin{aligned} F(r,\theta) &= \int M(r,\theta)\partial r + h(\theta) \\ &= \int (\theta^2+1)\cos r\,\partial r + h(\theta) \\ &= (\theta^2+1)\sin r + h(\theta) \end{aligned}$$

Then

$$\frac{\partial F(r,\theta)}{\partial\theta} = 2\theta\sin r + \frac{dh(\theta)}{d\theta}$$

But we must have

$$\frac{\partial F(r,\theta)}{\partial\theta} = N(r,\theta) = 2\theta\sin r.$$

Thus

$$2\theta\sin r = 2\theta\sin r + \frac{dh(\theta)}{d\theta} \Leftrightarrow 0 = \frac{dh(\theta)}{d\theta}.$$

Thus $h(\theta) = C_0$, where C_0 is an arbitrary constant, and so

$$F(r,\theta) = (\theta^2+1)\sin r + C_0.$$

Hence a one- parameter family of solution is $F(r,\theta) = C_1$, or

$$(\theta^2+1)\sin r + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$(\theta^2+1)\sin r = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(\theta^2+1)\cos r\,dr + 2\theta\sin r\,d\theta = 0$ is $(\theta^2+1)\sin r = C$.

7. $(y\sec^2 x + \sec x\tan x)dx + (\tan x + 2y)dy = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y) = y\sec^2 x + \sec x\tan x\,, N(x,y) = \tan x + 2y,$$
$$\frac{\partial M(x,y)}{\partial y} = \sec^2 x\,, \frac{\partial N(x,y)}{\partial x} = \sec^2 x.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = \sec^2 x$$

we can conclude that the equation $(y\sec^2 x + \sec x\tan x)dx + (\tan x + 2y)dy = 0$ is exact differential equation. Furthermore, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = y\sec^2 x + \sec x \tan x \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = \tan x + 2y.$$

From the first of these,

$$F(x,y) = \int M(x,y)\partial x + h(y)$$
$$= \int (y\sec^2 x + \sec x \tan x)\partial x + h(y)$$
$$= y\tan x + \sec x + h(y).$$

Then

$$\frac{\partial F(x,y)}{\partial y} = \tan x + \frac{dh(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \tan x + 2y.$$

Thus

$$\tan x + 2y = \tan x + \frac{dh(y)}{dy} \Leftrightarrow 2y = \frac{dh(y)}{dy}.$$

Thus $h(y) = y^2 + C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = y\tan x + \sec x + y^2 + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$y\tan x + \sec x + y^2 + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$y\tan x + \sec x + y^2 = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(y\sec^2 x + \sec x \tan x)dx + (\tan x + 2y)dy = 0$ is $y\tan x + \sec x + y^2 = C$.

8. $\left(\frac{x}{y^2} + x\right)dx + \left(\frac{x^2}{y^3} + y\right)dy = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y) = \frac{x}{y^2} + x, N(x,y) = \frac{x^2}{y^3} + y,$$
$$\frac{\partial M(x,y)}{\partial y} = -\frac{2x}{y^3}, \frac{\partial N(x,y)}{\partial x} = \frac{2x}{y^3}.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = -\frac{2x}{y^3} \neq \frac{\partial N(x,y)}{\partial x} = \frac{2x}{y^3}$$

we can conclude that the differential equation $\left(\frac{x}{y^2} + x\right)dx + \left(\frac{x^2}{y^3} + y\right)dy = 0$ is not exact (non- exact) differential equation.

9. $\left(\frac{2s-1}{t}\right)ds + \left(\frac{s-s^2}{t^2}\right)dt = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(s,t) = \frac{2s-1}{t}, N(s,t) = \frac{s-s^2}{t^2},$$
$$\frac{\partial M(s,t)}{\partial t} = \frac{1-2s}{t^2}, \frac{\partial N(s,t)}{\partial s} = \frac{1-2s}{t^2}.$$

Since

$$\frac{\partial M(s,t)}{\partial t} = \frac{\partial N(s,t)}{\partial s} = \frac{1-2s}{t^2}$$

we can conclude that the differential equation $\left(\frac{2s-1}{t}\right) ds + \left(\frac{s-s^2}{t^2}\right) dt = 0$ is exact differential equation. Furthermore, we must find F such that

$$\frac{\partial F(s,t)}{\partial s} = M(s,t) = \frac{2s-1}{t} \text{ and } \frac{\partial F(s,t)}{\partial t} = N(s,t) = \frac{s-s^2}{t^2}.$$

From the first of these,

$$\begin{aligned} F(s,t) &= \int M(s,t)\partial x + h(t) \\ &= \int \left(\frac{2s-1}{t}\right) \partial s + h(t) \\ &= \frac{s^2-s}{t} + h(t). \end{aligned}$$

Then

$$\frac{\partial F(s,t)}{\partial t} = -\frac{s^2}{t^2} + \frac{s}{t^2} + \frac{dh(t)}{dt} = \frac{s-s^2}{t^2} + \frac{dh(t)}{dt}$$

But we must have

$$\frac{\partial F(s,t)}{\partial t} = N(s,t) = \frac{s-s^2}{t^2}.$$

Thus

$$\frac{s-s^2}{t^2} = \frac{s-s^2}{t^2} + \frac{dh(t)}{dt} \Leftrightarrow 0 = \frac{dh(t)}{dt}.$$

Thus $h(t) = C_0$, where C_0 is an arbitrary constant, and so

$$F(s,t) = \frac{s^2-s}{t} + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$\frac{s^2-s}{t} + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$\frac{s^2-s}{t} = C$$

or we can write this as $s^2 - s = Ct$, where $C = C_1 - C_0$ is an arbitrary constant. So, we conclude that the general solution of the exact differential equation $\left(\frac{2s-1}{t}\right) ds + \left(\frac{s-s^2}{t^2}\right) dt = 0$ is $s^2 - s = Ct$.

10. $\left(\frac{2y^{\frac{3}{2}}+1}{x^{\frac{1}{2}}}\right) dx + \left(3x^{\frac{1}{2}}y^{\frac{1}{2}} - 1\right) dy = 0.$

Solution:

Our first duty is to determine whether or not the equation is exact or not. Here

$$M(x,y) = \frac{2y^{\frac{3}{2}}+1}{x^{\frac{1}{2}}}, N(x,y) = 3x^{\frac{1}{2}}y^{\frac{1}{2}} - 1,$$

$$\frac{\partial M(x,y)}{\partial y} = 3\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}, \frac{\partial N(x,y)}{\partial x} = \frac{3}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} = \frac{3}{2}\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = 3\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \neq \frac{\partial N(x,y)}{\partial x} = \frac{3}{2}\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

we can conclude that the differential equation $\left(\frac{2y^{\frac{3}{2}}+1}{x^{\frac{1}{2}}}\right)dx + \left(3x^{\frac{1}{2}}y^{\frac{1}{2}} - 1\right)dy = 0$ is not exact (non- exact) differential equation.

Part 2. (In each of the following equations determine the constant A such that the equation is exact, and solve the resulting exact equation).

a. $(x^2 + 3xy)dx + (Ax^2 + 4y)dy = 0.$

Solution:

Suppose

$$M(x,y) = x^2 + 3xy \text{ and } N(x,y) = Ax^2 + 4y.$$

Then, we obtain

$$\frac{\partial M(x,y)}{\partial y} = 3x \text{ and } \frac{\partial N(x,y)}{\partial x} = 2Ax.$$

In order to make the differential equation become exact differential equation, it must be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 3x.$$

Thus

$$2Ax = 3x \Leftrightarrow A = \frac{3}{2}.$$

Therefore, we obtain that the differential equation $(x^2 + 3xy)dx + \left(\frac{3}{2}x^2 + 4y\right)dy = 0$ is exact differential equation. Furthermore, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = x^2 + 3xy \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{3}{2}x^2 + 4y.$$

From the first of these,

$$F(x,y) = \int M(x,y)\partial x + h(y) = \int (x^2 + 3xy)\partial x + h(y) = \frac{1}{3}x^3 + \frac{3}{2}x^2 y + h(y).$$

Then

$$\frac{\partial F(x,y)}{\partial y} = \frac{3}{2}x^2 + \frac{dh(y)}{dy}.$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{3}{2}x^2 + 4y.$$

Thus

$$\frac{3}{2}x^2 + 4y = \frac{3}{2}x^2 + \frac{dh(y)}{dy} \Leftrightarrow 4y = \frac{dh(y)}{dy}.$$

Thus $h(y) = 2y^2 + C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = \frac{1}{3}x^3 + \frac{3}{2}x^2 y + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 y + 2y^2 + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 y + 2y^2 = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(x^2 + 3xy)dx + \left(\frac{3}{2}x^2 + 4y\right)dy = 0$ is $\frac{1}{3}x^3 + \frac{3}{2}x^2 y + 2y^2 = C$.

b. $\left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + \left(\frac{Ax+1}{y^3}\right)dy = 0.$

Solution:

Suppose

$$M(x,y) = \frac{1}{x^2} + \frac{1}{y^2} \text{ and } N(x,y) = \frac{Ax+1}{y^3}.$$

Then, we obtain

$$\frac{\partial M(x,y)}{\partial y} = -\frac{2}{y^3} \text{ and } \frac{\partial N(x,y)}{\partial x} = \frac{A}{y^3}.$$

In order to make the differential equation become exact differential equation, it must be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = -\frac{2}{y^3}.$$

Thus

$$\frac{A}{y^3} = -\frac{2}{y^3} \Leftrightarrow A = -2.$$

Therefore, we obtain that the differential equation $\left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + \left(\frac{-2x+1}{y^3}\right)dy = 0$ is exact differential equation. Furthermore, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = \frac{1}{x^2} + \frac{1}{y^2} \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{-2x+1}{y^3}.$$

From the first of these,

$$F(x,y) = \int M(x,y)\partial x + h(y) = \int \left(\frac{1}{x^2} + \frac{1}{y^2}\right)\partial x + h(y) = -\frac{1}{x} + \frac{x}{y^2} + h(y).$$

Then

$$\frac{\partial F(x,y)}{\partial y} = -\frac{2x}{y^3} + \frac{dh(y)}{dy}.$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{-2x+1}{y^3}.$$

Thus

$$\frac{-2x+1}{y^3} = -\frac{2x}{y^3} + \frac{dh(y)}{dy} \Leftrightarrow \frac{1}{y^3} = \frac{dh(y)}{dy}.$$

Thus $h(y) = -\frac{1}{2y^2} + C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = -\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $\left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + \left(\frac{-2x+1}{y^3}\right)dy = 0$ is $-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = C$.

Part 3. (Solve using grouping method)

1. $\frac{1}{x}dy - \frac{y}{x^2}dx = 0.$

Solution:

From $\frac{1}{x}dy - \frac{y}{x^2}dx = 0$ we obtain

$$d\left(\frac{y}{x}\right) = d(C).$$

So

$$\frac{y}{x} = C$$

is the general solution of the differential equation $\frac{1}{x}dy - \frac{y}{x^2}dx = 0.$

2. $2xy\frac{dy}{dx} + y^2 - 2x = 0.$

Solution:

$2xy\dfrac{dy}{dx} + y^2 - 2x = 0$

$\Leftrightarrow 2xy\dfrac{dy}{dx} = 2x - y^2$

$\Leftrightarrow (2x - y^2)dx - 2xydy = 0$

From $(2x - y^2)dx - 2xydy = 0$, we group the term as follows

$$2xdx - (y^2dx + 2xydy) = 0.$$

Thus

$$d(x^2) - d(xy^2) = d(C).$$

So,

$$x^2 - xy = C$$

is the general solution of the differential equation $2xy\frac{dy}{dx} + y^2 - 2x = 0.$

3. $2(y+1)e^x dx + 2(e^x - 2y)dy = 0.$

Solution:

From $2(y+1)e^x dx + 2(e^x - 2y)dy = 0$, we group the term as follows

$$(2ye^x dx + 2e^x dy) + 2e^x dx - 4ydy = 0.$$

Thus

$$d(2ye^x) + d(2e^x) - d(2y^2) = d(C).$$

Therefore,

$$2ye^x + 2e^x - 2y^2 = C$$

is the general solution of the differential equation

$$2(y+1)e^x dx + 2(e^x - 2y)dy = 0.$$

4. $(2xy + 6x)dx + (x^2 + 4y^3)dy = 0.$

Solution:

From $(2xy + 6x)dx + (x^2 + 4y^3)dy = 0$, we group the term as follows

$$(2xydx + x^2dy) + 6xdx + 4y^3dy = 0.$$

Thus

$$d(x^2y) + d(3x^2) + d(y^4) = d(C).$$

So,

$$x^2y + 3x^2 + y^4 = C$$

is the general solution of the differential equation

$$(2xy + 6x)dx + (x^2 + 4y^3)dy = 0.$$

Part 4. (Quiz)

1. Which of the following differential equations can be made exact by multiplying by x^2?

(a) $\frac{dy}{dx} + \frac{2}{x}y = 4.$

Solution:

$$\frac{dy}{dx} + \frac{2}{x}y = 4 \Leftrightarrow \frac{dy}{dx} = 4 - \frac{2}{x}y \Leftrightarrow \left(4 - \frac{2}{x}y\right)dx - dy = 0$$

By multiplying both sides by x^2, we obtain

$(4x^2 - 2xy)dx + (-x^2)dy = 0.$

Here

$$M(x,y) = 4x^2 - 2xy, N(x,y) = -x^2,$$
$$\frac{\partial M(x,y)}{\partial y} = -2x, \frac{\partial N(x,y)}{\partial x} = -2x$$

Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = -2x$$

we can conclude that the differential equation $(3x + 2y)dx + (2x + y)dy = 0$ is exact differential equation. In the other word, the differential equation $\frac{dy}{dx} + \frac{2}{x}y = 4$ can be made exact by multiplying by x^2.

(b) $x\frac{dy}{dx} + 3y = x^2.$

Solution:

$$x\frac{dy}{dx}+3y=x^2 \Leftrightarrow x\frac{dy}{dx}=x^2-3y \Leftrightarrow (x^2-3y)dx+(-x)dy=0.$$

By multiplying both sides by x^2, we obtain

$$(x^4-3x^2y)dx+(-x^3)dy=0.$$

Here

$$M(x,y)=x^4-3x^2y, N(x,y)=-x^3,$$
$$\frac{\partial M(x,y)}{\partial y}=-3x^2, \frac{\partial N(x,y)}{\partial x}=-3x^2.$$

Since

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}=-3x^2$$

we can conclude that the differential equation $(x^4-3x^2y)dx+(-x^3)dy=0$ is exact differential equation. In the other word, the differential equation $x\frac{dy}{dx}+3y=x^2$ can be made exact by multiplying by x^2.

(c) $\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=x.$

Solution:

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=x \Leftrightarrow \frac{1}{x}\frac{dy}{dx}=x+\frac{1}{x^2}y \Leftrightarrow \left(x+\frac{1}{x^2}y\right)dx+\left(-\frac{1}{x}\right)dy=0.$$

By multiplying both sides by x^2, we obtain

$$(x^3+y)dx+(-x)dy=0.$$

Here

$$M(x,y)=x^3+y, N(x,y)=-x,$$
$$\frac{\partial M(x,y)}{\partial y}=1, \frac{\partial N(x,y)}{\partial x}=-1.$$

Since

$$\frac{\partial M(x,y)}{\partial y}=1\neq\frac{\partial N(x,y)}{\partial x}=-1$$

we can conclude that the differential equation $(x^3+y)dx+(-x)dy=0$ is not exact differential equation. In the other word, the differential equation $\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=x$ can not be made exact by multiplying by x^2.

(d) $\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}y=3.$

Solution:

$$\frac{1}{x}\frac{dy}{dx}+\frac{1}{x^2}y=3 \Leftrightarrow \frac{1}{x}\frac{dy}{dx}=3-\frac{1}{x^2}y \Leftrightarrow \left(3-\frac{1}{x^2}y\right)dx+\left(-\frac{1}{x}\right)dy=0.$$

By multiplying both sides by x^2, we obtain

$$(3x^2-y)dx+(-x)dy=0.$$

Here

$$M(x,y)=3x^2-y, N(x,y)=-x,$$
$$\frac{\partial M(x,y)}{\partial y}=-1, \frac{\partial N(x,y)}{\partial x}=-1.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = -1$$

we can conclude that the differential equation $(3x^2 - y)dx + (-x)dy = 0$ is exact differential equation. In the other word, the differential equation $\frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 3$ can be made exact by multiplying by x^2.

2. Consider the differential equation

$$(4x + 3y^2)dx + 2xydy = 0.$$

(a) Show that this equation is not exact.

Proof. Here

$$M(x,y) = 4x + 3y^2, N(x,y) = 2xy,$$
$$\frac{\partial M(x,y)}{\partial y} = 6y, \frac{\partial N(x,y)}{\partial x} = 2y.$$

Since

$$\frac{\partial M(x,y)}{\partial y} = 6y \neq \frac{\partial N(x,y)}{\partial x} = 2y$$

we can conclude that the differential equation $(4x + 3y^2)dx + 2xydy = 0$ is is not exact differential equation. ■

(b) Find an integrating factor of the form x^n, where n is a positive integer.

Solution:

From (a) we know that the differential equation $(4x + 3y^2)dx + 2xydy = 0$ is not exact. But then, we can find an integrating factor $u(x,y) = x^n$, where n is a positive integer such that the differential equation $x^n(4x + 3y^2)dx + x^n(2xy)dy = 0$ is exact.

Assume that $(4x + 3y^2)dx + x^n(2xy)dy = 0$ is exact differential equation.

Here

$x^n(4x + 3y^2)dx + x^n(2xy)dy = 0$

$\Leftrightarrow (4x^{n+1} + 3x^n y^2)dx + (2x^{n+1}y)dy = 0.$

Let

$$M(x,y) = 4x^{n+1} + 3x^n y^2 \text{ and } N(x,y) = 2x^{n+1}y.$$

Then, we obtain

$$\frac{\partial M(x,y)}{\partial y} = 6x^n y \text{ and } \frac{\partial N(x,y)}{\partial x} = (2n+2)x^n y$$

Since $(4x + 3y^2)dx + x^n(2xy)dy = 0$ is exact differential equation, we must obtain

$$6x^n y = \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = (2n+2)x^n y,$$

that is

$$6x^n y = (2n+2)x^n y.$$

Hence $n = 2$.

Thus, the integrating factor of the form x^n, where n is a positive integer such that the differential equation $x^n(4x + 3y^2)dx + x^n(2xy)dy = 0$ is exact is x^2.

(c) Multiplying the given equation through by the integrating factor found in (b) and solve the resulting exact equation.

Solution:

From (b) we know that the differential equation $(4x^3 + 3x^2y^2)dx + (2x^3y)dy = 0$ is exact. Now, we will find the solution of this exact differential equation or in the other word we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 4x^3 + 3x^2y^2 \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^3y.$$

From the first of these,

$$\begin{aligned} F(x,y) &= \int M(x,y)\partial x + h(y) \\ &= \int (4x^3 + 3x^2y^2\ )\partial x + h(y) \\ &= x^4 + x^3y^2 + h(y). \end{aligned}$$

Then

$$\frac{\partial F(x,y)}{\partial y} = 2x^3y + \frac{dh(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^3y.$$

Thus

$$2x^3y = 2x^3y + \frac{dh(y)}{dy} \Leftrightarrow 0 = \frac{dh(y)}{dy}.$$

Thus $h(y) = C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = x^4 + x^3y^2 + C_0.$$

Hence a one- parameter family of solution is $F(x,y) = C_1$, or

$$x^4 + x^3y^2 + C_0 = C_1$$

Combining the constsnts C_0 and C_1 we may write this solution as

$$x^4 + x^3y^2 = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation $(4x^3 + 3x^2y^2)dx + (2x^3y)dy = 0$ is $x^4 + x^3y^2 = C$.